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Twist four longitudinal structure function for a positronium-like bound state in light-front QED

Asmita Mukherjee

Saha Institute of Nuclear Physics, 1/AF, Bidhan Nagar, Calcutta 700064, India

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Abstract

To have an analytic understanding of the higher-twist structure functions, we calculate twist four longitudinal structure function for a positronium-like bound state in weak coupling light-front QED. We find that in the weakly coupled system, the fermionic part of F_L is related to the kinetic energy of the fermions and not to the interaction. We verify a previously proposed sum rule in this limit, which in this case reduces to a relation connecting the kinetic and the potential energies to the binding energy of positronium. Using the analytic form of the wave function of positronium in this limit, we show that the constituent counting rule does not hold for $x \to 1$. The twist four F_L in this limit is similar in form to a widely used phenomenological ansatz.

Keywords: Light-front Hamiltonian; Bound states; Twist four; Structure function

1. Introduction

Higher twist or power suppressed contributions to deep inelastic scattering structure functions involve nontrivial nonperturbative information about the structure of hadrons. Recent experimental results indicate that these higher twist effects play an important role in the kinematical range of the SLAC experiments and it is important to have a clear physical picture of them. Light-front Hamiltonian QCD offers a theoretical tool to investigate the deep inelastic scattering structure functions. This is based on physical intuitions and at the same time employs well defined field theoretical calculational techniques. The structure functions are expressed as the Fourier transform of the matrix elements of light-front bilocal currents. Fock space expansion of the target state allows us to express these in terms of light-front multiparton wave functions. The interesting aspect of this formulation is that both perturbative and nonperturbative issues can be addressed within the same framework [1]. Recently, the twist four part of the longitudinal structure function F_L and the transverse polarized structure function g_T, which is a twist three contribution have been analyzed in this approach and various interesting issues associated with them have been addressed [2,3]. The structure functions can be calculated once the light-front bound state wave functions are known. However, the actual calculation for a QCD bound state in $3+1$ dimension is highly complicated and

E-mail address: asmita@theory.saha.ernet.in (A. Mukherjee).

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it requires the recently developed similarity renormalization techniques [4] for the light-front QCD Hamiltonian. The spectra of the heavy quark bound states like charmonium and bottomonium have been investigated using this technique [5]. The analytic form of the wave function is not obtained so far. This is because, the similarity renormalization technique generates an additional confining interaction in the effective Hamiltonian which makes it impossible to solve the effective Hamiltonian analytically even in the leading order in bound state Hamiltonian perturbation theory. In this work, we have performed a much simpler but quite interesting analysis. We have calculated the twist-four part of the longitudinal structure function for a positronium-like bound state in light-front QED in the weak coupling limit. In fact, in this limit, QCD results are not expected to differ too much from the QED results. As a result, this analysis is important since it tests and illustrates the approach in QCD. The advantage here is that, in the leading order in bound state perturbation theory, the bound state equation can be solved analytically and the wave function is known. This allows an analytic understanding of the problem.

In the weak coupling limit, we show that our previously proposed sum rule [2] reduces to a relation connecting the kinetic and potential energies to the binding energy of positronium. Using the analytic wave function of positronium in this limit, we show that the structure functions fall faster than that predicted by constituent counting rule near $x \to 1$. Also, we find that the twist four F_L in the weak coupling limit has a form which is similar to a widely used phenomenological formula, which connects the twist four distribution to the twist two distribution.

2. Twist four longitudinal structure function

We consider a positronium-like bound state $|P\rangle$ given by

$$\begin{aligned}
|P\rangle = {} & \sum_{\sigma_1,\sigma_2} \int \frac{dk_1^+\, d^2k_1^\perp}{\sqrt{2(2\pi)^3 k_1^+}} \int \frac{dk_2^+\, d^2k_2^\perp}{\sqrt{2(2\pi)^3 k_2^+}} \phi_2(P|k_1,\sigma_1;k_2,\sigma_2)\sqrt{2(2\pi)^3 P^+}\,\delta^3(P-k_1-k_2) \\
& \times b^\dagger(k_1,\sigma_1) d^\dagger(k_2,\sigma_2)|0\rangle \\
& + \sum_{\sigma_1,\sigma_2,\lambda_3} \int \frac{dk_1^+\, d^2k_1^\perp}{\sqrt{2(2\pi)^3 k_1^+}} \int \frac{dk_2^+\, d^2k_2^\perp}{\sqrt{2(2\pi)^3 k_2^+}} \int \frac{dk_3^+\, d^2k_3^\perp}{\sqrt{2(2\pi)^3 k_3^+}} \phi_3(P|k_1,\sigma_1;k_2,\sigma_2;k_3,\lambda_3)\sqrt{2(2\pi)^3 P^+} \\
& \times \delta^3(P-k_1-k_2-k_3) b^\dagger(k_1,\sigma_1) d^\dagger(k_2,\sigma_2) a^\dagger(k_3,\lambda_3)|0\rangle. \qquad (1)
\end{aligned}$$

Here ϕ_2 is the probability amplitude to find an electron and positron in the positronium, ϕ_3 is the probability amplitude to find an electron, positron and a photon in the positronium. We consider up to three particle sector. We introduce Jacobi momenta $(x, \kappa^\perp)$ and the boost invariant amplitudes, ψ_2 and ψ_3 [6].

We calculate $F_4^{\tau=4}$ in light-front QED in light front gauge, $A^+ = 0$ using weak coupling approximation. This includes the effect of dynamical photon. It can be shown that, for a weak coupling theory the results are the same as obtained using a nonrelativistic approximation. However, the entire calculation is fully relativistic and exact in the leading order in bound state perturbation theory [7].

The twist-4 part of the fermionic component of the longitudinal structure function is given by

$$F_{L(f)}^{\tau=4}(x) = M_1 + M_2, \qquad (2)$$

$$M_1 = \frac{1}{Q^2}\frac{x^2(P^+)^2}{2\pi}\int dy^-\, e^{-\frac{i}{2}P^+y^-x}\langle P|\overline{\psi}(y^-)\gamma^-\psi(0) - \overline{\psi}(0)\gamma^-\psi(y^-)|P\rangle, \qquad (3)$$

$$M_2 = -\frac{(P^\perp)^2}{(P^+)^2}\frac{1}{Q^2}\frac{x^2(P^+)^2}{2\pi}\int dy^-\, e^{-\frac{i}{2}P^+y^-x}\langle P|\overline{\psi}(y^-)\gamma^+\psi(0) - \overline{\psi}(0)\gamma^+\psi(y^-)|P\rangle. \qquad (4)$$

We shall take the mass of the state $|P\rangle$ to be M and the electron and positron mass to be m. In the weak coupling (nonrelativistic) limit the helicity dependence of the wave function factorizes away, so it is sufficient to consider one helicity sector. Here we shall take the two particle state with helicities σ_1 and σ_2 up.

For a positronium state $|P\rangle$ given by Eq. (1)we obtain,

$$
\begin{aligned}
F_{L(f)}^{\gamma=4}(x)_{\text{diag}} &= (M_1)_{\text{diag}} + (M_2)_{\text{diag}} \\
&= \frac{4}{Q^2}\int dx_1\, d^2\kappa_1^\perp \left(\left(\kappa_1^\perp\right)^2 + m^2\right)|\psi_2|^2\left[\delta(x-x_1) + \delta(1-x-x_1)\right] \\
&\quad + \frac{4}{Q^2}\sum\int dx_1\, d^2\kappa_1^\perp \int dx_2\, d^2\kappa_2^\perp\, |\psi_3|^2 \\
&\quad \times \left[\left(\left(\kappa_1^\perp\right)^2 + m^2\right)\delta(x-x_1) + \left(\left(\kappa_2^\perp\right)^2 + m^2\right)\delta(x-x_2)\right].
\end{aligned}
\tag{5}
$$

The off-diagonal contributions to $F_{L(f)}^{\tau=4}(x)$ comes from M_1 alone

$$
(M_1)_{\text{off-diag}} = -\frac{4}{Q^2}\frac{e^2}{2(2\pi)^3}\int dx_1\, d^2\kappa_1^\perp \int dy\, d^2\kappa^\perp\left[M_1^a + M_1^b + M_1^c + M_1^d\right],
\tag{6}
$$

where

$$
M_1^a = \frac{1}{(x_1-y)^2 E}\left[\frac{2(\kappa_1^\perp)^2 y}{x_1}\delta(x-x_1) - \frac{2(\kappa^\perp)^2 x_1}{y}\delta(x-y)\right]\left|\psi_2^{\sigma_1\sigma_2}\left(x_1,\kappa_1^\perp\right)\right|^2,
\tag{7}
$$

$$
M_1^b = \frac{1}{E(x-y)^2}\left[-\left(\kappa_1^\perp\right)^2\delta(x-x_1) + \left(\kappa^\perp\right)^2\delta(x-y)\right]\left(\psi_2^{*\sigma_1\sigma_2}\left(x_1,\kappa_1^\perp\right)\psi_2^{\sigma_1\sigma_2}\left(y,\kappa^\perp\right) + \text{h.c.}\right),
\tag{8}
$$

$$
M_1^c = \frac{1}{(y-x_1)^2 E'}\left[\frac{2(\kappa_1^\perp)^2 y}{(1-x_1)}\delta(1-x-x_1) - \frac{2(\kappa^\perp)^2(1-x_1)}{y}\delta(x-1+y)\right]\left|\psi_2^{\sigma_1\sigma_2}\left(x_1,\kappa_1^\perp\right)\right|^2,
\tag{9}
$$

$$
M_1^d = \frac{1}{E'(y-x_1)}\left[\left(\kappa^\perp\right)^2\delta(x+y-1) - \left(\kappa_1^\perp\right)^2\delta(1-x-x_1)\right]\left(\psi_2^{*\sigma_1\sigma_2}\left(x_1,\kappa_1^\perp\right)\psi_2^{\sigma_1\sigma_2}\left(y,\kappa^\perp\right) + \text{h.c.}\right).
\tag{10}
$$

In this calculation we have taken all operators to be normal ordered. Also, in the Eq. (7)–(10) we have neglected all mass terms in the vertex, since these terms are suppressed in the nonrelativistic limit. The energy denominators are given by

$$
\begin{aligned}
E &= M^2 - \frac{(\kappa^\perp)^2 + m^2}{y} - \frac{(\kappa_1^\perp)^2 + m^2}{1-x_1} - \frac{(\kappa_1^\perp - \kappa^\perp)^2}{x_1 - y}, \\
E' &= M^2 - \frac{(\kappa_1^\perp)^2 + m^2}{x_1} - \frac{(\kappa^\perp)^2 + m^2}{y} - \frac{(\kappa_1^\perp - \kappa^\perp)^2}{(y-x_1)}.
\end{aligned}
\tag{11}
$$

We define the twist four longitudinal photon structure function as [2]

$$
\begin{aligned}
F_{L(g)}^{\tau=4}(x) &= \frac{1}{Q^2}\frac{xP^+}{2\pi}\int dy^-\, e^{-\frac{i}{2}P^+y^-x} \\
&\quad \times \bigg[\langle P|(-)F^{+\lambda}(y^-)F_\lambda^-(0) + \frac{1}{4}g^{+-}F^{\lambda\sigma}(y^-)F_{\lambda\sigma}(0)|P\rangle \\
&\quad - \frac{(P^\perp)^2}{(P^+)^2}\langle P|F^{+\lambda}(y^-)F_\lambda^+(0)|P\rangle + (y^- - 0)\bigg].
\end{aligned}
\tag{12}
$$

$F_{L(g)}^{\tau=4}(x)$ has both diagonal and off-diagonal parts. We take all operators to be normal ordered and we get

$$
\begin{aligned}
\frac{F_{L(g)}^{\tau=4}(x)}{x}_{\text{diag}} &= \frac{4}{Q^2}\int dx_1\, d^2\kappa_1^\perp \int dy\, d^2\kappa^\perp\, |\psi_3|^2\frac{(-\kappa_1^\perp - \kappa^\perp)^2}{(1-x_1-y)}\delta(1-x_1-x-y) \\
&\quad - \frac{4}{Q^2}\frac{4e^2}{2(2\pi)^3}\int dx_1\, d^2\kappa_1^\perp \int dy\, d^2\kappa^\perp \psi_2^*\left(x_1,\kappa_1^\perp\right)\psi_2\left(y,\kappa^\perp\right)\frac{1}{(x_1-y)^2}\delta(x_1-x-y).
\end{aligned}
\tag{13}
$$

The second term in the right hand side is the contribution of the instantaneous interaction. The off-diagonal contribution is

$$F^{\tau=4}_{L(g)}(x)_{\text{off-diag}} = G_1 + G_2, \tag{14}$$

where

$$\begin{aligned}
G_1 = &-\frac{4}{Q^2}\frac{e^2}{2(2\pi)^3}\int dx_1\, d^2\kappa_1^\perp \int dy\, d^2\kappa^\perp \frac{x}{E(x_1-y)^2} \\
&\times\bigg[\bigg(-4\frac{(\kappa_1^\perp-\kappa^\perp)^2}{(x_1-y)} + \frac{2(\kappa_1^\perp)^2}{x_1} - \frac{2(\kappa^\perp)^2}{y}\bigg)\big|\psi_2^{\sigma_1\sigma_2}\big(x_1,\kappa_1^\perp\big)\big|^2 \\
&+\bigg(\frac{2(\kappa_1^\perp-\kappa^\perp)^2}{(x_1-y)} + \frac{(\kappa_1^\perp)^2}{(1-x_1)} - \frac{(\kappa^\perp)^2}{(1-y)}\bigg)\Big(\psi_2^{*\sigma_1\sigma_2}\big(x_1,\kappa_1^\perp\big)\psi_2^{\sigma_1\sigma_2}\big(y,\kappa^\perp\big) + \text{h.c.}\Big)\bigg]\delta(x-x_1+y),
\end{aligned} \tag{15}$$

$$\begin{aligned}
G_2 = &-\frac{4}{Q^2}\frac{e^2}{2(2\pi)^3}\int dx_1\, d^2\kappa_1^\perp \int dy\, d^2\kappa^\perp \frac{x}{E'(y-x_1)^2} \\
&\times\bigg[\bigg(-\frac{4(\kappa_1^\perp-\kappa^\perp)^2}{(y-x_1)} - \frac{2(\kappa^\perp)^2}{(1-y)} + \frac{2(\kappa_1^\perp)^2}{(1-x_1)}\bigg)\big|\psi_2^{\sigma_1\sigma_2}\big(x_1,\kappa_1^\perp\big)\big|^2 \\
&+\bigg(-\frac{2(\kappa_1^\perp-\kappa^\perp)^2}{(y-x_1)} - \frac{(\kappa^\perp)^2}{y} + \frac{(\kappa_1^\perp)^2}{x_1}\bigg)\Big(\psi_2^{*\sigma_1\sigma_2}\big(x_1,\kappa_1^\perp\big)\psi_2^{\sigma_1\sigma_2}\big(y,\kappa^\perp\big) + \text{h.c.}\Big)\bigg]\delta(x_1+x-y),
\end{aligned} \tag{16}$$

where E and E' are given by Eq. (11).

From these expressions, we calculate

$$\begin{aligned}
&\int_0^1 \frac{F^{\tau=4}_{L(q)}(x) + F^{\tau=4}_{L(g)}(x)}{x}\,dx \\
&= \frac{4}{Q^2}\int dx\, d^2\kappa^\perp\, \psi_2{}^*\psi_2\bigg[\frac{(\kappa^\perp)^2}{x} + \frac{(\kappa^\perp)^2}{1-x}\bigg] \\
&\quad + \frac{4}{Q^2}\sum\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp\, \psi_3{}^*\psi_3\bigg[\frac{(\kappa^\perp)^2}{x} + \frac{(q^\perp)^2}{y} + \frac{(-\kappa^\perp-q^\perp)^2}{(1-x-y)}\bigg] \\
&\quad + \frac{4}{Q^2}\frac{e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp\, [A_1+A_2+B_1+B_2] \\
&\quad - \frac{4}{Q^2}\frac{4e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp \frac{1}{(x-y)^2}\psi_2^*\big(x,\kappa^\perp\big)\psi_2\big(y,q^\perp\big),
\end{aligned} \tag{17}$$

where

$$A_1 = \frac{1}{E}\frac{1}{(x-y)}\Big[V_1\big|\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\big|^2 + V_2\psi_2^{*\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\psi_2^{\sigma_1\sigma_2}\big(y,q^\perp\big)\Big], \tag{18}$$

$$A_2 = \frac{1}{E'}\frac{1}{(y-x)}\Big[V_2'\psi_2^{*\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\psi_2^{\sigma_1\sigma_2}\big(y,q^\perp\big) + V_1'\big|\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\big|^2\Big], \tag{19}$$

$$B_1 = \frac{1}{E}\frac{1}{(x-y)}\Big[V_1\big|\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\big|^2 + V_2\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\psi_2^{*\sigma_1\sigma_2}\big(y,q^\perp\big)\Big], \tag{20}$$

$$B_2 = \frac{1}{E'}\frac{1}{(y-x)}\Big[V_2'\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\psi_2^{*\sigma_1\sigma_2}\big(y,q^\perp\big) + V_1'\big|\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\big|^2\Big], \tag{21}$$

where V_1, V_2, V_1' and V_2' are given by

$$V_1 = \left[\frac{2(\kappa^\perp - q^\perp)^2}{(x-y)^2} + \frac{(x+y)}{(x-y)}\left(\frac{(\kappa^\perp)^2}{x^2} + \frac{(q^\perp)^2}{y^2}\right)\right], \tag{22}$$

$$V_2 = V_2' = \left[\frac{(\kappa^\perp)^2(1-2x)}{x(1-x)(x-y)} + \frac{(q^\perp)^2(2y-1)}{y(1-y)(x-y)} - \frac{2(\kappa^\perp - q^\perp)^2}{(x-y)^2}\right], \tag{23}$$

$$V_1' = \left[\frac{2(\kappa^\perp - q^\perp)^2}{(x-y)^2} + \frac{(2-x-y)}{(y-x)}\left(\frac{(\kappa^\perp)^2}{(1-x)^2} + \frac{(q^\perp)^2}{(1-y)^2}\right)\right]. \tag{24}$$

In these expressions we have kept only those terms in the vertex which survive in the nonrelativistic limit. The helicities σ_1 and σ_2 are both up and we have neglected all mass terms in the vertex since in this limit they are suppressed. Also in the weak coupling (nonrelativistic) limit, we consider only photon exchange interactions and the off-diagonal terms proportional to $|\psi_2|^2$ originating from self energy effects can be neglected.

Now, from the expression of ψ_3 (see [6]) we can write

$$\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp\, |\psi_3|^2 = \frac{e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp \left[\frac{1}{E}B_1 + \frac{1}{E'}B_2\right], \tag{25}$$

where B_1 and B_2 are given earlier. Using this, one can write the second term in the right-hand side of Eq. (17) as

$$\frac{4}{Q^2}\frac{e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp \left[\frac{1}{E}B_1 + \frac{1}{E'}B_2\right]\left[\frac{(\kappa^\perp)^2 + m^2}{x} + \frac{(q^\perp)^2 + m^2}{y} + \frac{(-\kappa^\perp - q^\perp)^2}{(1-x-y)}\right]. \tag{26}$$

Considering the fact that the total energy is conserved, one can write this as

$$-\frac{4}{Q^2}\frac{e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp (B_1 + B_2). \tag{27}$$

So we get

$$\begin{aligned}
&\int_0^1 \frac{F_{L(q)}^{\tau=4}(x) + F_{L(g)}^{\tau=4}(x)}{x}\, dx \\
&\quad = \frac{4}{Q^2}\int dx\, d^2\kappa^\perp\, {\psi_2}^*\psi_2\left[\frac{(\kappa^\perp)^2}{x} + \frac{(\kappa^\perp)^2}{1-x}\right] + \frac{4}{Q^2}\frac{e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp [A_1 + A_2] \\
&\qquad - \frac{4}{Q^2}\frac{4e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp \int dy\, d^2q^\perp \frac{1}{(x-y)^2}\psi_2^*\big(x, \kappa^\perp, 1-x, -\kappa^\perp\big)\psi_2\big(y, q^\perp\big).
\end{aligned} \tag{28}$$

Also, if we denote $\frac{(\kappa^\perp)^2 + m^2}{x(1-x)}$ by M_0^2, then in the nonrelativistic limit, it can be shown that, $M^2 - M_0^2 \simeq O(e^4)$. So we neglect this difference in the energy denominators and replace the bound state mass in E and E' by $M^2 = \frac{(\kappa^\perp)^2 + m^2}{x(1-x)}$. The energy denominators then become

$$E = \frac{(\kappa^\perp)^2 + m^2}{x} - \frac{(q^\perp)^2 + m^2}{y} - \frac{(\kappa^\perp - q^\perp)^2}{x-y} = -\frac{1}{(x-y)}\left[\left(\frac{m}{x}\right)^2(x-y)^2 + \big(\kappa^\perp - q^\perp\big)^2\right], \tag{29}$$

$$E' = \frac{(\kappa^\perp)^2 + m^2}{1-x} - \frac{(q^\perp)^2 + m^2}{1-y} + \frac{(\kappa^\perp - q^\perp)^2}{x-y} = \frac{1}{(x-y)}\left[\left(\frac{m}{1-x}\right)^2(x-y)^2 + \big(\kappa^\perp - q^\perp\big)^2\right]. \tag{30}$$

We get, in this limit,

$$\int_0^1 \frac{F_L^{\tau=4}(x)}{x}\, dx = \frac{4}{Q^2}\int dx\, d^2\kappa^\perp\, |\psi_2|^2 \frac{(\kappa^\perp)^2 + m^2}{x(1-x)}$$

$$
\begin{aligned}
& -\frac{4}{Q^2}\frac{2e^2}{2(2\pi)^3}\int dy\, d^2q^\perp\, \psi_2^*\big(x,\kappa^\perp,1-x,-\kappa^\perp\big)\psi_2\big(y,q^\perp,1-y,-q^\perp\big) \\
& \times\left[\left(\frac{m}{x}\right)^2\frac{1}{(\kappa^\perp-q^\perp)^2+(\frac{m}{x})^2(x-y)^2}\left(\frac{m}{(1-x)}\right)^2\frac{1}{(\kappa^\perp-q^\perp)^2+(\frac{m}{(1-x)})^2(x-y)^2}\right].
\end{aligned} \tag{31}
$$

Here in the weak coupling limit we have omitted the spin indices.

The Fermionic part of the Hamiltonian density is given by

$$
\theta_f^{+-}=i\overline{\psi}\gamma^-\partial^+\psi=2\psi^{+\dagger}\big[\alpha^\perp.\big(i\partial^\perp+gA^\perp\big)+\gamma^0m\big]\frac{1}{i\partial^+}\big[\alpha^\perp.\big(i\partial^\perp+gA^\perp\big)+\gamma^0m\big]\psi^+. \tag{32}
$$

The gauge bosonic part of the Hamiltonian density is given by

$$
\begin{aligned}
\theta_g^{+-} &= -F^{+\lambda}F_\lambda^-+\frac{1}{4}g^{+-}(F_{\lambda\sigma})^2=\frac{1}{4}\big(\partial^+A^-\big)^2+\frac{1}{2}F^{ij}F_{ij} \\
&= \big(\partial^iA^j\big)^2+2e\partial^iA^i\left(\frac{1}{\partial^+}\right)2\big(\psi^+\big)^\dagger\psi^++e^2\left(\frac{1}{\partial^+}\right)2\big(\psi^+\big)^\dagger\psi^+\left(\frac{1}{\partial^+}\right)2\big(\psi^+\big)^\dagger\psi^+.
\end{aligned} \tag{33}
$$

The fermionic part of the longitudinal momentum density is given by $\theta_f^{++}=i\overline{\psi}\gamma^+\partial^+\psi$ and the gauge bosonic part of the longitudinal momentum density $\theta_g^{++}=-F^{+\lambda}F_\lambda^-$. For a positronium-like bound state, we calculate the matrix element of θ_f^{+-} and θ_g^{+-}. The matrix elements have both diagonal and off-diagonal contribution. The diagonal contribution to the matrix element from the fermionic and the gauge bosonic part is given by

$$
\begin{aligned}
& \left[\langle P|\theta^{+-}|P\rangle-\frac{(P^\perp)^2}{(P^+)^2}\langle P|\theta^{++}|P\rangle\right]_{\text{diag}} \\
&\quad = 2\int dx_1\, d^2\kappa_1^\perp\, \psi_2^*\psi_2\left\{\frac{(\kappa_1^\perp)^2}{x_1}+\frac{(\kappa_2^\perp)^2}{(1-x_1)}\right\} \\
&\qquad +2\int dx_1\, d^2\kappa_1^\perp\int dx_2\, d^2\kappa_2^\perp\, \psi_3^*\psi_3\left\{\frac{(\kappa_1^\perp)^2}{x_1}+\frac{(\kappa_2^\perp)}{x_2}+\frac{(-\kappa_1^\perp-\kappa_2^\perp)^2}{(1-x_1-x_2)}\right\} \\
&\qquad -\frac{8e^2}{2(2\pi)^3}\int dx_1\, d^2\kappa_1^\perp\int dy\, d^2\kappa^\perp\, \psi_2^*(x_1,\kappa_1^\perp)\psi_2\big(y,\kappa^\perp\big)\frac{1}{(x-y)^2},
\end{aligned} \tag{34}
$$

where $\theta^{+-}=\theta_f^{+-}+\theta_g^{+-}$.

The off-diagonal part can be written as

$$
\left[\langle P|\theta^{+-}|P\rangle-\frac{(P^\perp)^2}{(P^+)^2}\langle P|\theta^{++}|P\rangle\right]_{\text{off-diag}}=\mathcal{V}_1+\mathcal{V}_2, \tag{35}
$$

where

$$
\begin{aligned}
\mathcal{V}_1 &= \frac{2e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp\int dy\, d^2q^\perp\,\frac{1}{E}\frac{1}{(x-y)} \\
&\quad\times\left[2V_1\big|\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\big|^2+V_2\Big(\psi_2^{*\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\psi_2^{\sigma_1\sigma_2}\big(y,q^\perp\big)+\text{h.c.}\Big)\right],
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\mathcal{V}_2 &= \frac{2e^2}{2(2\pi)^3}\int dx\, d^2\kappa^\perp\int dy\, d^2q^\perp\,\frac{1}{E'}\frac{1}{(y-x)} \\
&\quad\times\left[2V_1'\big|\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp\big)\big|^2+V_2'\Big(\psi_2^{\sigma_1\sigma_2}\big(x,\kappa^\perp,1-x,-\kappa^\perp\big)\psi_2^{\sigma_1\sigma_2}\big(y,q^\perp\big)+\text{h.c.}\Big)\right],
\end{aligned} \tag{37}
$$

where the expressions for V_1, V_2, V_1', V_2', E and E' are given earlier. As before, we have taken the two particle state with both σ_1, σ_2 up.

Considering only the photon exchange interactions and putting $M^2 = M_0^2$ in the energy denominators as before, one obtains in the nonrelativistic limit for a weak coupling theory,

$$\begin{aligned}
&\left[\langle P|\theta^{+-}|P\rangle - \frac{(P^\perp)^2}{(P^+)^2}\langle P|\theta^{++}|P\rangle\right] \\
&\quad = 2\int dx\, d^2\kappa^\perp |\psi_2|^2 \frac{(\kappa^\perp)^2 + m^2}{x(1-x)} - 2\frac{2e^2}{2(2\pi)^3}\int dy\, d^2q^\perp\, \psi_2^*\big(x,\kappa^\perp,1-x,-\kappa^\perp\big)\psi_2\big(y,q^\perp,1-y,-q^\perp\big) \\
&\qquad \times \left[\left(\frac{m}{x}\right)^2 + \frac{1}{(\kappa^\perp - q^\perp)^2 + (m/x)^2(x-y)^2} + \left(\frac{m}{(1-x)}\right)^2 \frac{1}{(\kappa^\perp - q^\perp)^2(m/(1-x))^2(x-y)^2}\right].
\end{aligned} \tag{38}$$

We can see that the right-hand side of the above equation, which is nothing but Coulomb interaction, has exactly the same form as the interaction part of $\int_0^1 \frac{F_L^{\tau=4}(x)}{x}$. So, the interaction part of $\int_0^1 \frac{F_L^{\tau=4}(x)}{x}\,dx$ can be related to the expectation value of the Coulomb interaction. Since $\kappa^\perp$ and $q^\perp$ are small in the nonrelativistic limit, all $(\kappa^\perp)^2$ and $(q^\perp)^2$ dependence in the numerator of the interaction terms are neglected compared to the m^2 dependent terms. However, the term proportional to $(\kappa^\perp - q^\perp)^2/(x-y)^2$ cannot be neglected because both x and y are almost equal and this term cancels the instantaneous interaction in the nonrelativistic limit. Both of these terms originate from $F_{L(g)}$ and one can see that only the gauge bosonic part of the longitudinal structure function is important for the Coulomb interaction in the weak coupling limit. The m^2 terms in the energy denominators combine with the other terms to give nonvanishing contribution. The fermionic part only gives contribution to the kinetic energy of the fermions. Reminding oneself that we are working in the light-front gauge and not in the Coulomb gauge, this is a manifestation of the gauge invariance of the separation of the Hamiltonian density into a fermionic and a gauge bosonic part.

We introduce the three-vector $\vec{p}$ as $\vec{p} = (\kappa, \kappa_z)$, where κ_z is defined through a coordinate transformation from $x \in [0,1]$ to $\kappa_z \in [-\infty, \infty]$ by $x \equiv 1/2 + \kappa_z/(2\sqrt{\kappa^{\perp 2} + \kappa_z^2 + m^2})$. We introduce the bound state wave function $\phi(\vec{p})$ which is normalized as $\int d^3\vec{p}\,\phi^*(\vec{p})\phi(\vec{p}) = 1$. The bound state equation for the positronium in the weak coupling limit can be written as [8]

$$\Big[M^2 - 4\big((\vec{p})^2 + m^2\big)\Big]\phi(\vec{p}) = -\frac{2e^2}{2(2\pi)^3}\int d^3\vec{p}\,\phi(\vec{p}\,')\frac{4m}{(\vec{p} - \vec{p}\,')^2}. \tag{39}$$

We write Eq. (38) as

$$\begin{aligned}
&\left[\langle P|\theta^{+-}|P\rangle - \frac{(P^\perp)^2}{(P^+)^2}\langle P|\theta^{++}|P\rangle\right] \\
&\quad = 2\int d^3\vec{p}\,|\phi(\vec{p})|^2 4\big[(\vec{p})^2 + m^2\big] - \frac{4e^2}{2(2\pi)^3}\int d^3\vec{p}\int d^3\vec{p}\,'\phi^*(\vec{p})\,\phi(\vec{p}\,')\frac{4m}{(\vec{p} - \vec{p}\,')^2}.
\end{aligned} \tag{40}$$

We can now see the more familiar form of the Coulomb interaction. Multiplying the bound state equation (39) by $\phi^*(\vec{p})$ and integrating we get

$$M^2 = \int d^3\vec{p}\,|\phi(\vec{p})|^2 4\big[(\vec{p})^2 + m^2\big] - \frac{2e^2}{2(2\pi)^3}\int d^3\vec{p}\int d^3\vec{p}\,'\,\phi^*(\vec{p})\,\phi(\vec{p}\,')\frac{4m}{(\vec{p} - \vec{p}\,')^2}. \tag{41}$$

Hence, from Eqs. (31), (38) and (41) it can be seen that the sum rule [2] is satisfied in the weak coupling limit for a positronium target in light-front QED and it can be written as

$$\int_0^1 \frac{F_L^{\tau=4}(x)}{x}\,dx = \frac{2}{Q^2}\left[\langle P|\theta^{+-}|P\rangle - \frac{(P^\perp)^2}{(P^+)^2}\langle P|\theta^{++}|P\rangle\right] = 4\frac{M^2}{Q^2}. \tag{42}$$

In the nonrelativistic limit for a weak coupling theory $M^2 = 4m^2 + 4mB_e$, where B_e is the binding energy of positronium.

From Eq. (41) we obtain,

$$B_e = \int d^3\vec{p}\,|\phi(\vec{p})|^2 \frac{(\vec{p})^2}{m} - \frac{2e^2}{2(2\pi)^3} \int d^3\vec{p} \int d^3\vec{p}\,'\,\phi^*(\vec{p})\phi(\vec{p}\,') \frac{1}{(\vec{p}-\vec{p}\,')^2}. \tag{43}$$

The first term in the right-hand side is the kinetic energy with $m/2$ being the reduced mass of the two body system and the second term is the expectation value of the Coulomb interaction. So we see that in the weak coupling limit, the sum rule reduces to a relation connecting the kinetic and potential energies to the binding energy.

3. $F_L^{\tau=4}$ for the ground state of positronium

The bound state equation (39) can be analytically solved for QED, which is the primary motivation for studying QED. The ground state wave function of positronium is given by

$$\phi_{\nu,s_e,s_{e'}}(\vec{p}, s, s') = \phi_\nu(\vec{p})\delta_{s_e,s}\delta_{s_{e'},s'}, \tag{44}$$

where s_e and $s_{e'}$ label the spin quantum numbers of the electron and positron, respectively, and ν denotes all the other quantum numbers $\nu = n, l, m$ correspond with the standard nonrelativistic quantum numbers of hydrogen. The spin part factorizes out and the wave function is normalized to 1. The wave function is given by

$$\phi_\nu(\vec{p}) = \frac{4(e_n)^{5/2}}{((e_n)^2 + (\vec{p})^2)^2} Y_\nu(\Omega_p), \tag{45}$$

where $e_n = m\alpha/(2n)$ and $Y_\nu(\Omega_p) = Y_{n,l,m}(\Omega)$ are Hyperspherical harmonics. Here $0 \leqslant |m| \leqslant l \leqslant n-1$.

For 1s state of positronium, we have $Y_{1,0,0} = 1/\sqrt{2\pi^2}$. In terms of x and $\kappa^\perp$, the 1s state wave function can be written as

$$\phi_2(x, \kappa^\perp) = \sqrt{\frac{m}{\pi^2}} \frac{4(e_1)^{5/2}}{\left[(e_1)^2 - m^2 + \frac{1}{4}\frac{(\kappa^\perp)^2 + m^2}{x(1-x)}\right]^2} \tag{46}$$

which agrees with [9] for nonrelativistic $x \simeq 1/2$.

The weak coupling limit contributions to the structure functions $F_2(x)$ and $F_L^{\tau=4}(x)$ can be directly evaluated using this wave function

$$F_2(x) = \int d^2\kappa^\perp \left|\phi_2(x, \kappa^\perp)\right|^2 = \int d^2\kappa^\perp \frac{Ax^4(1-x)^4}{\left[m^2[(1-2x)^2 + \alpha^2 x(1-x)] + (\kappa^\perp)^2\right]^4}, \tag{47}$$

where $A = 4.78 \times 10^{-12} (\mathrm{MeV})^6$. The integral is convergent and can be evaluated analytically introducing a cutoff Λ. We get,

$$F_2(x) = 28.15 \times 10^{-11} \frac{x^4(1-x)^4}{\left[(1-2x)^2 + \alpha^2 x(1-x)\right]^3}. \tag{48}$$

$F_2(x)$ is very sharply peaked at $x = 1/2$.

The twist four longitudinal structure function is given by

$$\frac{F_L^{\tau=4}}{x} = \frac{4}{Q^2} \int d^2\kappa^\perp \frac{(\kappa^\perp)^2}{x(1-x)} |\phi_2|^2 = \frac{4}{Q^2} \int d^2\kappa^\perp \frac{A(\kappa^\perp)^2 x^3(1-x)^3}{\left[m^2[(1-2x)^2 + \alpha^2 x(1-x)] + (\kappa^\perp)^2\right]^4}. \tag{49}$$

Here we have considered only the $(\kappa^\perp)^2$ dependent part, the integral of which is directly connected to the kinetic energy of the electron–positron pair. The above form is similar to the widely used phenomenological ansatz for the twist four distribution [10] but not exactly the same. Evaluating the integral analytically, we get

$$\frac{F_L(x)}{x} = 14.7 \times 10^{-11} \frac{1}{Q^2} \frac{x^3(1-x)^3}{\left[(1-2x)^2 + \alpha^2 x(1-x)\right]^2}. \tag{50}$$

$Q^2 F_L(x)/x$ is maximum at $x = 1/2$ and it is also sharply peaked. The structure function calculated above (48) falls faster than that expected from constituent counting rule near $x \to 1$ [11]. This is because the nonrelativistic wave function behaves differently in the asymptotic region. It can be seen that it behaves as $1/(k^\perp)^4$ for large $k^\perp$ whereas the relativistic wave function behaves as $1/(k^\perp)^2$ [12]. The asymptotic behavior of the form factor of positronium $F(Q^2)$ can be calculated analytically using the above wave function. It can be shown that $F(Q^2) \sim 1/Q^4$ when Q^2 is large, as expected for positronium in the nonrelativistic limit [12]. This means that it does not have a monopole behavior in this limit and one cannot expect the constituent counting rule to hold.

To summarize, in this work, we have investigated the twist-four longitudinal structure function for a positronium-like bound state in light-front QED in the weak coupling limit. In this limit, we get expressions that look similar to the familiar nonrelativistic expressions, but the entire calculation is fully relativistic in the leading order in light-front bound state perturbation theory. We have explicitly verified a sum rule for $F_L^{\tau=4}$ that we previously proposed. It is worth mentioning here that in the nonperturbative context, we have investigated before the twist four longitudinal structure function for a meson in $1+1$ dimension and showed that the sum rule is satisfied using 't Hooft's equation. In this work, we have shown that in the weak coupling limit, the sum rule reduces to a relation connecting the kinetic and the potential energies to the binding energy of positronium. We have also shown that, in this limit, the fermionic part of $F_L^{\tau=4}$ contributes only to the kinetic energy of the fermions and not to the interactions. The twist four F_L in this limit looks similar to a much used phenomenological ansatz, however, here we get the result directly from field theory. The structure function falls faster than that predicted by constituent counting rule, as expected.

The twist four part of the longitudinal structure function is important since it is the leading nonperturbative contribution to F_L. The leading twist contribution to F_L is perturbative, in contrast to the case of F_2. This analysis for a bound state in weak-coupling light-front QED is quite interesting since it gives an idea of what goes in such a calculation in light-front QCD. Similarity renormalization up to $O(e^2)$ does not produce any additional interaction in the effective QED Hamiltonian and in the weak coupling limit, one can work with the canonical Hamiltonian. The overall computational framework in QCD is the same and this analysis in light-front QED allows an analytic understanding of the problem.

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